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EXTREMAL 3-CONNECTED GRAPHS

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Let G be a 3-connected graph with minimum degree at least d and at least $2d$ vertices. For any three distinct vertices x, y, z there is a path from x to z through y and having length at least $2d-2$. In this paper, we characterize those graphs for which no such path has length exceeding $2d-2$.

1. Introduction

All graphs considered in this paper are simple. For basic graph-theoretic terminology, we refer the reader to [2].

The following was proved in [4, 5].

Theorem 1. *Let G be a 3-connected graph with minimum degree at least d and at least $2d-1$ vertices, for some positive integer d. Then, for any three vertices x, y, and z in G, there is an (x, z)-path of length at least $2d-2$ which contains y.*

The purpose of this paper is to examine the extremal graphs for this result. Not surprisingly, these extremal graphs are also in the class of graphs shown to be extremal for a similar Ore-type result of Enomoto [3]. We shall assume that the reader is familiar with the proofs given in [4]. The necessary theorems and lemmas will be restated here.

2. Definitions

Let G be a graph. We denote the set of vertices of G by $V(G)$, and the set of edges of G by $E(G)$. For any vertex x in G, the set of neighbours of x is $N(x)$, and the degree of x is $d(x)$. Let P be a path in G. For vertices x and z on P, we denote the section of P (or its reversal) from x to z and including both x and z by $P[x, z]$. If P has not been previously defined, we use $P[x, z]$ to denote any path from x to z. Let x and z be vertices of G, and let Y be a subset of the vertices of G. An (x, Y, z)*-path* is an (x, z)-path P which includes every vertex of Y. When Y has exactly one vertex y, P will also be referred to as an (x, y, z)-path. An $(x, Y, z:d)$*-path* is an (x, Y, z)-path whose length is at least d. If Y is empty, we shall refer to P as an $(x, z:d)$-path.

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We now state, without proof, two of the lemmas from [4].

Lemma 1. *Let x and z be vertices of a graph G. Suppose that G has at least one (x, z)-path, and let P be a longest such path. Furthermore, suppose that there exist vertices u and v adjacent to P, but not on P, and a vertex q such that:*

(i) $|N(u) \cap N(v) \cap V(P)| = r$;

(ii) $|(N(u) \cup N(v)) \cap V(P)| = k$, *for some* $k \geq 2$, *and*

(iii) *there is a* $(u, q, v : l)$*-path* L *disjoint from* P.

Then G contains an $(x, q, z : 2(k-1) + sl)$-path, where

$$s = \begin{cases} 1, & \text{if } 0 \leq r \leq 1, \\ r-1, & \text{if } 2 \leq r \leq k. \end{cases}$$

Lemma 2. *Let x and z be vertices of a graph G. Suppose that G has at least one (x, z)-path, and let P be a longest such path. Suppose that there exist vertices u_1, u_2, and u_3 adjacent to P, but not on P, and a vertex q such that:*

(i) $|(N(u_1) \cup N(u_2) \cup N(u_3)) \cap V(P)| = k$, *for some* $k \geq 3$,

(ii) *if u_m and u_n both have exactly one neighbour on P, then these two neighbours are distinct, for $m, n \in \{1, 2, 3\}$ and $m \neq n$, and*

(iii) *there is a $(u_m, q, u_n : l)$-path P_{mn} disjoint from P, for $m, n \in \{1, 2, 3\}$ and $m \neq n$.*

Then G has an $(x, q, z : 2k + 2l - 2)$-path.

In [5], we gave the following two extremal examples for Theorem 1. We shall prove that these are the only extremal examples for Theorem 1 having more than $2d - 1$ vertices.

Example 1. Let H be any graph on d vertices, and let S be an independent set of vertices disjoint from H, $|S| \geq d - 1$. Let G be the graph obtained by adding edges from every vertex of S to every vertex of H.

Example 2. Let H be the disjoint union of complete graphs each having $d - 2$ vertices. Let $\{x, z, v\}$ be three vertices disjoint from H, and let F be any graph with $V(F) = \{x, v, z\}$. Let G be the graph obtained by joining every vertex of H to every vertex of F.

Lemma 3 characterizes the extremal graphs for a result similar to that of Lovász [6, Exercise 10.19].

Lemma 3. *Let G be a 2-connected graph and let x, y and z be distinct vertices of G. Suppose that every vertex of G, except, possibly, x and one of y or z, has degree at least d. Then, for any vertex q, there is an $(x, q, y : d)$-path. Furthermore,* *if G has at least $d+2$ vertices, then there is either an $(x, q, y:d+1)$-path or an $(x, q, z:d+1)$-path.*

Proof. If G has $d+1$ vertices, then G is K_{d+1} or $K_{d+1}-e$, for $e=xy$ or $e=xz$, and it is not difficult to find $(x, q, y:d)$-paths. Hence, we may assume that G has at least $d+2$ vertices.

If $q=x$, we may pick any vertex q' in $V(G)-\{x, y, z\}$ and replace q by q'. Any (x, q', w)-path will contain q. We may thus assume that $x \neq q$. We may also assume, without loss of generality, that $y \neq q$.

We shall proceed by induction on d and $|V(G)|$. Suppose $d=2$. Suppose, also, that q is not adjacent to x. By Menger's Theorem [7], there are internally-disjoint paths $P[x, q]$ and $Q[q, y]$. Then, PQ is an $(x, q, y:3)$-path. Hence, we may assume that q is adjacent to both x and y, and also that $q=z$ or q is adjacent to z.

Suppose that q is adjacent to z. Thus, by Menger's Theorem, there are disjoint paths P, Q from $\{x, y\}$ to $\{q, z\}$. Then, $P \cup Q \cup \{qz\}$ is an $(x, q, y:3)$-path.

Now suppose that $z=q$. There is some vertex v in $V(G)-\{x, y, z\}$ which is adjacent to one of y or z. If v is adjacent to z, then there are disjoint paths P, Q from $\{x, y\}$ to $\{v, z\}$ and $P \cup Q \cup \{vz\}$ is an $(x, q, y:3)$-path. The case in which v is adjacent to y is similar. Thus, the result is true for $d=2$.

We may thus assume that $d>2$. Let $G'=G-x$. Then, every vertex of G', except, possibly, one of y or z, has degree at least $d-1$.

Suppose that G' is 2-connected. If $N(x)=\{y, z\}$, then by induction there is a $(y, q, z:d)$-path in G', and hence an $(x, q, z:d+1)$-path in G. Thus we may assume that there is some vertex x' in $N(x)-\{y, z\}$. By induction, there is an $(x', q, y:d-1)$-path in G' and, hence an $(x, q, y:d)$-path in G.

If $|V(G)|>d+1$, then $|V(G')|>d$. Hence, there is either an $(x', q, y:d)$-path in G' or an $(x', q, z:d)$-path in G'. Thus, there is either an $(x, q, y:d+1)$-path in G or an $(x, q, z:d+1)$-path in G.

We may therefore assume that G' is separable. Choose an endblock B of G' with cutvertex b, so that y is not in $B-b$ and there is either (i) a (b, q, y)-path Q in G', or (ii) a (q, b, y)-path Q' in G'.

Let x' be any neighbour of x in $B-b$. In case (i), there is an $(x', b:d-1)$-path P in B, and $R=xx'PQ$ is an $(x, q, y:d)$-path in G. In case (ii), there is an $(x', q, b:d-1)$-path P' in B, and $R'=xx'P'Q'[b, y]$ is an $(x, q, y:d)$-path in G. In both cases, the path has length at least $d+1$ unless $b=y$ and q is in B (and case (ii) applies).

Now, if z is not in B, then R' followed by any (b, z)-path is an $(x, q, z:d+1)$-path. Finally, if z is in B, and hence in $B-b$, then choose any other endblock B' with cutvertex b'. Let x'' be any neighbour of x in $B'-b'$. There is an $(x'', b':d-1)$-path P'' in B', and a (b', q, z)-path Q'' in G'. Then, $xx''P''Q''$ is an $(x, q, z:d+1)$-path in G.

Therefore, in all cases, we have constructed paths of the desired types. □

Bondy and Jackson [1] prove that under the conditions of Lemma 3, some pair of vertices chosen from $\{x, y, z\}$ is connected by a path of length at least $\min\{2d-2, n-1\}$, where n is the number of vertices of the graph. We shall have need of a pair of paths, each of length at least $d+1$, which together meet all three vertices.

3. 3-connected graphs

We shall proceed in the same manner as in [4]. We begin with a 3-connected graph G with minimum degree d and a triple of vertices x, y, z. Throughout the remainder of this discussion, we shall assume that G has at least $2d$ vertices but no $(x, y, z: 2d-1)$-path. Choose a longest (x, z)-path P. Label the vertices of P, as they appear along P, $x = x_1, x_2, \ldots, x_e = z$, $e \geqslant 2d-1$. If y is not on P, let $y' = y$. Otherwise, choose y' to be any vertex not on P. Let H be the component of $G - V(P)$ containing y'.

In each of the four cases encountered in the proof of Theorem 1, we characterize the component H and its vertices of attachment. Then we prove that P has length exactly $2d-2$. Hence we may assume that y is on P. Therefore, we have characterized all components of $G - V(P)$ and their vertices of attachment. Thus, we have characterized G.

Case 1. H has a single vertex

All the conditions of Lemma 1 are satisfied by setting $u = v = q = y'$, $l = 0$, and $r = d(y') \geqslant d$. If $N(y') \neq \{x_{2i-1} \mid i = 1, 2, \ldots, \frac{1}{2}(e+1)\}$, then there is an $(x, y', z: 2d-1)$-path. Thus, we know H, its vertices of attachment, and that P has length exactly $2d-2$.

Case 2. $V(H) = \{y', w\}$, for some vertex w

The conditions of Lemma 1 are satisfied by setting $u = q = y'$, $v = w$, $l = 1$, $k \geqslant d(y') - 1 \geqslant d-1$, $k \geqslant 3$, and $r \geqslant 2(d-1) - k$. By Lemma 1, there is an (x, y', z)-path P' such that

$$\begin{aligned}|E(P')| &\geqslant 2(k-1) + s \geqslant 2(k-1) + r - 1\\ &\geqslant k - 2 + 2(d-1) - 1 \geqslant 2(d-1) + (k-3)\\ &\geqslant 2(d-1).\end{aligned}$$

We note that $|E(P')| = 2d-2$ only if $k = 3$. Hence $d \leqslant 4$. Also, $r = 2(d-1) - k$.

If $d = 3$, then $r = 1$, $s = 1$, and hence $s > r - 1$. Therefore $|E(P')| > 2d-2$.

If $d = 4$, then $r = 3$, $s = 2$, $N(y') = N(w) = \{x_1, x_4, x_7\}$. Again, we have identified H, its vertices of attachment, and determined that the length of P is exactly $2d-2$.

Case 3. H is separable

As in [4], we can choose a pair of vertices u, v, with d_1, d_2 neighbours, respectively, on P, and a $(u, y', v:2d - d_1 - d_2)$-path R. The conditions of Lemma 1 are satisfied with $l = 2d - d_1 - d_2$, $k \geq \max\{d_1, d_2\}$. Hence, G, has an (x, y', z)-path P', with

$$|E(P')| \geq 2(k-1) + sl \geq 2d - 2.$$

Here, equality can only hold if $k = d_1 = d_2$, $N(u) = N(v)$, $r = k \geq 2$, and $s = 1$, $0 \leq r \leq 2$. Thus $r = k = 2$, and $N(u) = N(v) = \{x, z\}$.

By the choice of u and v, no other internal vertex of the endblocks containing these two vertices is adjacent to any other vertex of P. Since G is 3-connected, there are three disjoint paths from R to P. Without loss of generality, two of these paths are ux and vz. Let the third path be $R'[w, x_i]$, $w \in V(R)$. Since $|E(R)| = 2(d-2)$, we note that w is the cutvertex separating the endblocks of H containing u and v and both of $R[u, w]$ and $R[w, v]$ have length exactly $d - 2$. Without loss of generality, y' is on $R[u, w]$.

If $i \geq d$, then

$$P'' = P[x, x_i]x_iwR[w, u]uz$$

is an $(x, y', z:2d-1)$-path. Similarly, if $i \leq d$, we can obtain another $(x, y', z:2d-1)$-path. Thus, for all choices of i, we can construct such a path, and there can be no separable components.

Case 4. H is 2-connected and has at least three vertices

As in [4], we choose vertices u_1, u_2, and u_3, satisfying conditions (i) and (ii) of Lemma 2, with d_1, d_2, and d_3 neighbours on P, respectively, such that $d_1 \geq d_2 \geq d_3$ and d_1 is maximum. We may also choose $(u_i, y', u_j:d - d_1)$-paths, P_{ij}, $i \neq j$, in H. If H has at least $d - d_1 + 2$ vertices, then, by Lemma 3, at least two of the three paths P_{ij} have length greater than $d - d_1$. Thus, the conditions of Lemma 2 are satisfied with $k \geq d_1$, and $l \geq d - d_1$. By Lemma 2, there is an (x, y', z)-path P', with

$$|E(P')| \; [illegible] + 2l \; [illegible] \geq 2d - 2.$$

Equality can hold only if $k = d_1$, $l = d - d_1$, and at least two of the paths P_{ij} have length exactly l. Therefore, H has exactly $d - d_1 + 1$ vertices and must be complete. Also, $d_1 = d_2 = d_3$ and $N(u_1) = N(u_2) = N(u_3)$.

Any two neighbours of u_1 on P must be separated by at least $d - d_1 + 2$ edges of P. Let x_i and x_j be two such neighbours of u_1, with $i < j$, and with no neighbour of u_1 between them on P. Let

$$P' = P[x, x_i]x_iu_1P_{12}u_2x_jP[x_j, z].$$

Since $d_1 \geq 3$,

$$|E(P')| \geq 2(d - d_1 + 2) + 2(d_1 - 3) \geq 2d - 2,$$

with equality only if $d_1 = 3$, the neighbours of u_1 on P are x, z and x_d, and P has length $2d-2$. In fact, every vertex of H has these same three neighbours on P, so we have identified H, its vertices of attachment, and ascertained that the length of P is $2d-2$.

We have completed the first part of our task. Since P has length exactly $2d-2$, we may assume that y is on P. Hence we may pick any vertex we choose as y'. This allows us to completely characterize the possible components that remain upon deletion of $V(P)$. Each component is either an isolated vertex or a K_{d-2}. Suppose that H and F are such components, where H is an isolated vertex and F is a K_{d-2}.

Let $V(H) = \{u\}$, let v, w be any two vertices of F, and let Q be any $(v, w: d-3)$-path in F. Then

$$P'' = P[x, x_{d-1}]x_{d-1}ux_{d+1}x_dvQwz$$

is an $(x, z: 2d-1)$-path, contradicting the choice of P. Therefore, either all components of $G - V(P)$ are isolated vertices or all are isomorphic to K_{d-2}.

Suppose that every component of $G - V(P)$ is an isolated vertex. We need to prove that $\{x_{2i} \mid i = 1, 2, \ldots, d-1\}$ is an independent set. Suppose, to the contrary, that $x_{2i}x_{2j}$ is an edge of G, with $i<j$. Let v be any vertex of $G - V(P)$. Then

$$P' = P[x, x_{2i-1}]x_{2i-1}vx_{2j-1}P[x_{2j-1}, x_{2i}]x_{2i}x_{2j}P[x_{2j}, z]$$

is an $(x, y, z: 2d-1)$-path. Hence, $\{x_{2i-1} \mid i = 1, 2, \ldots, d\}$ covers every edge of G. Thus, G is one of the graphs described in Example 1.

Now suppose that every component of $G - V(P)$ is isomorphic to K_{d-2}, and let H be one such component. Let u, v be vertices in H, and let Q be a $(u, v: d-3)$-path in H. Suppose that x_i is adjacent to x_j, for $1<i<d<j< 2d-1$. Let

$$P' = P[x, x_i]x_ix_jP[x_j, x_d]x_duQvz,$$

and

$$P'' = P[z, x_j]x_jx_iP[x_i, x_d]x_duQvx.$$

Then,

$$|E(P')| + |E(P'')| = |E(P)| + 2d,$$

contradicting the choice of P. Hence, $G' = G[\{x_2, \ldots, x_{d-1}\}]$ and $G'' = G[\{x_{d+1}, \ldots, x_{2d-2}\}]$ are also isomorphic to K_{d-2} and every vertex of G' and G'' is adjacent to each of x, z and x_d. Therefore, G is one of the graphs described in Example 2.

We have thus proven

Theorem 2. *Let G be a 3-connected graph with minimum degree at least d and with at least $2d-1$ vertices. Let x, y, and z be any three vertices of G. Then there is an $(x, y, z: 2d-2)$-path in G. Furthermore, suppose that G has at least $2d$*

vertices, that no set of exactly d vertices meets every edge, and that for every vertex $v \neq x, z$, $G - \{x, v, z\}$ *has a component which is not* K_{d-2}. *Then there is an* $(x, y, z : 2d - 1)$*-path in* G.

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